Augmented Happy Functions of Higher Powers

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Introduction

The *happy function*, *S*, squares the base 10 digits of a positive integer and sums them together. A positive integer, *a*, is called a *happy number* if there exists some $k \in \mathbf{Z}^+$ such that $S^k(a) = 1$ [7, 8]. Previous work has been done on the happy function [2, 5], the happy function with different integer powers [4], the happy function with different base number systems [3, 6], and constants added to the final sum of integers[1].

Let $T : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ be defined by taking the sum of some power of the digits of the integer plus a constant. Here, *T* is generalized to a two-parameter function and the results of applying these functions iteratively is studied.
Definition 1. For $c \geq 0$ and $q \geq 2$, let $T_{[c,q]} : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ be defined by

$$T_{[c,q]}\left(\sum_{i=0}^{n} a_i 10^i\right) = c + \sum_{i=0}^{n} {a_i}^q$$

for $0 \leq a_i \leq 9$ and a_n 6= 0.
This is referred to as an *augmented happy function with a higher power*. Let a *fixed point* be defined as an integer $a \in \mathbf{Z}^+$ such that $T_{[c,q]}(a) = a$ for some fixed *c* and *q*. Here, fixed points of $T_{[c,q]}$ are examined in decimal and arbitrary bases.

Fixed Point Characteristics

Table 1 lists fixed points of $T_{[c,3]}$ for small values of *c*, Table 2 lists fixed points of $T_{[c,4]}$ for small values of *c*, Table 3 lists fixed points of $T_{[c,5]}$ for small values of *c*, Table 4 lists fixed points of $T_{[c,6]}$ for small values of *c*, and Table 5 lists fixed points of $T_{[c,7]}$ for small values of *c*.

Table 1: Fixed Points for q=3 and $0 \leq c \leq 12$.

c	Fixed Points
0	1, 153, 370, 371, 407
1	
2	
3	12, 30, 31, 437, 474, 856
4	
5	
6	22, 1079
7	
8	
9	10, 11, 417, 566, 872, 962
10	
11	
12	20, 21, 263, 427, 865

Table 2: Fixed Points for q=4 and $0 \leq c \leq 12$.

c	Fixed Points
0	1, 1634, 8208, 9474
1	
2	2626
3	340, 341, 2565
4	20, 21, 8228
5	
6	
7	2616
8	
9	10, 11, 1643, 8218, 19949
10	
11	
12	9166

Table 3: Fixed Points for q=5 and $0 \leq c \leq 12$.

c	Fixed Points
0	1, 4150, 4151, 54748, 92727,93084, 194979
1	
2	
3	92338
4	
5	
6	17332
7	
8	
9	10, 11, 17303
10	
11	
12	4225, 79025, 92707

Table 4: Fixed Points for q=6 and $0 \leq c \leq 12$.

c	Fixed Points
0	1, 548834
1	528757, 629643, 688722, 715469
2	
3	4100, 4101
4	63015, 325286
5	
6	
7	141475
8	
9	10, 11, 548843, 774940, 774941
10	
11	381327
12	4110, 4111, 32055

Table 5: Fixed Points for q=7 and $0 \leq c \leq 12$.

c	Fixed Points
0	1, 1741725, 4210818, 9800817, 9926315, 14459929
1	
2	
3	1727530, 1727531, 5019535, 19973994
4	
5	
6	
7	
8	
9	10, 11, 8778478
10	
11	
12	2381603

Note that when q is an odd number, fixed points are only possible when c is a multiple of 3. The following two lemmas are used to prove if a is a fixed point and q is odd, then c must be a multiple of 3.

Lemma 1. If q is a positive odd integer and $0 \leq a_i \in Z$, then $a_i \equiv a^q{}_i$ (mod 3).
Proof. Given q is a positive odd integer, q takes the form $q = 2p + 1$, where $p \in \mathbf{Z}^+$. We consider the three cases where $a_i \equiv 0,1,2$ (mod 3). Since $0^q = 0$, if $0 \equiv a_i$ (mod 3), then $0 \equiv a^q{}_i$ (mod 3). Since $1^q = 1$, if $1 \equiv a_i$ (mod 3), $1 \equiv a^q{}_i$ (mod 3) . Since $2^q = 2^{2p+1} = 2 \cdot 4^p$ and $4 \equiv 1$ (mod 3), $2 \equiv 2^q$ (mod 3). Therefore if q is a positive odd integer, $a_i \equiv a^q{}_i$ (mod 3). □

Lemma 2. For all i such that $i \in \mathbf{Z}^+$, $1 \equiv 10^i$ (mod 3).
Proof. Since $1 \equiv 10$ (mod 3), any power of 10 must also be congruent to 1 modulo 3 since $1^i = 1$. Therefore, for all i such that $i \in \mathbf{Z}^+$, $1 \equiv 10^i$ (mod 3). □

Theorem 1. If a is a fixed point for given c and q such that q is an odd number, then c must be a multiple of three.
Proof. From Lemma 2, note that:

$$\sum_{i=0}^{n} 10^i a_i \equiv \sum_{i=0}^{n} a_i \quad (mod\ 3).$$

For a fixed a, note that

$$\sum_{i=0}^{n} 10^i a_i = c + \sum_{i=0}^{n} {a_i}^q.$$

This means that

$$\sum_{i=0}^{n} a_i \equiv c + \sum_{i=0}^{n} {a_i}^q \quad (mod\ 3).$$

As proven in Lemma 1, $a_i \equiv {a_i}^q \quad (mod\ 3)$ for all odd q. So

$$\sum_{i=0}^{n} a_i \equiv c + \sum_{i=0}^{n} a_i \quad (mod\ 3)$$

$$0 \equiv \quad (mod\ 3).$$

Therefore if a is a fixed point and q is an odd number, then c must be a multiple of three. □
Notice that this does not guarantee a fixed point at every odd q with a multiple of 3 for c. For example, $c = 6$ and $q = 7$ is one combination which contains no fixed points but fits the condition.
Definition 2. A set of generalized values for c and q that dictate whether or not a fixed point can exist under said c and q is called a "restricting pattern."

Theorem 2. The restricting pattern described in Theorem 1 is the only restricting pattern for $T_{[c,q]}(a)$.
Proof. In order for a restricting pattern to exist, c must be a multiple of some $k \in \mathbf{Z}^+$ such that $10^i \equiv 1 \pmod k$ for all $i \in \mathbf{Z}^+$. The only positive integers for which this is true are 3 and 9, so if another restricting pattern exists, c must be a multiple of 9. If a restricting pattern exists such that c is a multiple of 9, then for all $a_i \in \{0,1,2,...,9\}$, $a_i \equiv a^q{}_i \pmod 9$ must be true. If q is even, then q takes the form $q = 2m$ for $m \in Z^+$ and we have

$$\begin{aligned} 3^{2m} &\equiv (3^2)^m \quad (mod\ 9) \\ &\equiv 9^m \quad (mod\ 9) \\ &\equiv 0 \quad (mod\ 9). \end{aligned}$$

So if c is a multiple of 9 and a restricting pattern exists, q cannot be even. Similarly, if q is odd, then $q = 2m + 1$ and

$$\begin{aligned} 3^{2m+1} &\equiv (3^2)^m \times 3 \quad (mod\ 3) \\ &\equiv 9^m \times 3 \quad (mod\ 9) \\ &\equiv 0 \quad (mod\ 9). \end{aligned}$$

So if c is a multiple of 9 and a restricting pattern exists, q cannot be odd. Thus a restricting pattern cannot exist if c is a multiple of 9. Therefore, the restricting pattern described in Theorem 1 is the only restricting pattern for $T_{[c,q]}(a)$. □

For each paring of c and q, there is a value of a at and above which $T_{[c,q]}(a) < a$. The following lemmas describe this value when $q = 3$ and $q = 4$.

Lemma 3. Given $c \geq 0$, there exists a constant m such that for each $a \geq 10^m$, $T_{[c,3]}(a) < a$. This holds in particular for any $m \in \mathbf{Z}^+$ such that $10^m > 1360 + c$.
Proof. For a fixed a, note that

$$\begin{aligned} a - T_{[c,3]}(a) &= \sum_{i=0}^{n} 10^i a_i - c - \sum_{i=0}^{n} a_i{}^3 \\ &= \sum_{i=0}^{n} a_i\left(10^i - a_i{}^2\right) - c \\ &= a_n\left(10^i - a_n{}^2\right) + \sum_{i=2}^{n-1} a_i\left(10^i - a_i{}^2\right) + a_1(10^1 - a_1{}^2) + a_0(10^0 - a_0{}^2) - c \\ &\geq 1(10^m - 1) + 0 + 9(10 - 9^2) + 9(1 - 9^2) - c \\ &= 10^m - 1360 - c \\ &> 0. \end{aligned}$$

Therefore, $T_{[c,3]}(a) < a$ for all $a \geq 10^m$. □

Lemma 4. Given $c \geq 0$, there exists a constant m such that for each $a \geq 10^m$, $T_{[c,4]}(a) < a$. This holds in particular for any $m \in \mathbf{Z}^+$ such that $10^m > 18684 + c$.
Proof. For a fixed a, note that

$$\begin{aligned} a - T_{[c,4]}(a) &= \sum_{i=0}^{n} 10^i a_i - c - \sum_{i=0}^{n} a_i{}^4 \\ &= \sum_{i=0}^{n} a_i\left(10^i - a_i{}^3\right) - c \\ &= a_n\left(10^i - a_n{}^3\right) + \sum_{i=3}^{n-1} a_i\left(10^i - a_i{}^3\right) + a_2(10^2 - 3) + a_1(10^1 - a_1{}^3) + a_0(10^0 - a_0{}^3) - c \\ &\geq 1(10^m - 1) + 0 + 9(100 - 9^3) + 9(10 - 9^3) + 9(1 - 9^3) - c \\ &= 10^m - 18684 - c \\ &> 0. \end{aligned}$$

Therefore, $T_{[c,4]}(a) < a$ for all $a \leq 10^m$. □

The proofs for the values of a such that $T_{[c,q]}(a) < a$ for other values of q follow similarly. There is no general formula for any given c and q. The following lemmas show additional characteristics of fixed points.

Lemma 5. If a positive integer a is a fixed point for a given c and q and a ends in zero, then $a + 1$ is also a fixed point.
Proof. If a ends in zero and is a fixed point for a given c and q, then

$$\sum_{i=1}^{n} 10^i a_i = \sum_{i=0}^{n} 10^i a_i = c + \sum_{i=0}^{n} a_i{}^q = c + \sum_{i=1}^{n} a_i{}^q.$$

So

$$1+\sum_{i=0}^{n} 10^i a_i = 1 + c + \sum_{i=0}^{n} {a_i}^q.$$

Therefore, if a positive integer a is a fixed point for a given c and q and a ends in zero, then $a + 1$ is also a fixed point. □

Lemma 6. If $a_i \in \{0,1\}$ for all i, then there exists a c such that $T_{[c,q]}(a) = a$ for all q. The value of c is $a - \sum_{i=0}^{n} a_i$.
Proof. If $a_i \in \{0,1\}$ and $a_n \neq 0$ then

$$\begin{aligned} 0 &= a - T_{[c,q]}(a) \\ &= \sum_{i=0}^{n} 10^i a_i - c - \sum_{i=0}^{n} {a_i}^q \\ &= -c + \sum_{i=0}^{n} 10^i a_i - \sum_{i=0}^{n} a_i. \end{aligned}$$

Therefore, if $a_i \in \{0,1\}$ for all i, then there exists a c such that $T_{[c,q]}(a) = a$ for all q. The value of c is $a - \sum_{i=0}^{n} a_i$.

Fixed Points in Base b
Let b be the base for the positive integer a. Then the *augmented happy function of higher power in base b* is

$$T_{[c,q,b]}\left(\sum_{i=0}^{n} a_i b^i\right) = c + \sum_{i=0}^{n} {a_i}^q$$

for $0 \le a_i \le b - 1$ and $a_n 6= 0$. In base b, fixed points and restricting patterns are also possible. Below are a few examples.

Fixed points for $b = 4$, $q = 2$ and $0 \le c \le 33$.

c	Fixed Points
0	1
1	12, 32
2	22
3	10, 11, 30, 31
10	20,21
11	
12	
13	
20	
21	103
22	
23	
30	113, 133
31	102, 103
32	
33	100, 101

Fixed Points for b=16, q=2 and $0 \le$ b $\le$ D.

c	Fixed Points
0	1
1	
2	
3	14, F4
4	68, A8
5	
6	47, C7
7	78, 98
8	25, 88, E5
9	13, 36, D6, F3
A	
B	
C	
D	12, 57, B7, F2

Fixed points for $b = 4$, $q = 3$ and $0 \le c \le 33$.

c	Fixed Points
0	1, 20, 21, 130, 131, 203, 223, 313, 332
1	
2	
3	10, 11, 213, 232
10	
11	
12	330, 331
13	
20	
21	102, 122, 230, 231
22	
23	
30	112
31	
32	
33	100, 101, 120, 121, 302, 322

Fixed Points for b=16, q=3 and 0 ≤ b ≤ 1A

c	Fixed Points
0	1, 23, 40, 41, 156, 173, 208, 248, 285, 4A5, 5B0, 5B1, 60B, 64B, 8C0, 8C1, 99A, AA9, AC3, CA8, E69, EA0, EA1, 15EE
1	
2	
3	
4	
5	
6	378
7	
8	
9	12, 479, 76B, A7B, F67
A	
B	
C	
D	
E	
F	11, 32, 165, 218, 61B, 699, 9A9, 9C3
10	
11	
12	22, 172, AC2
13	
14	
15	30, 31, 238, 309, 349, 387, 63B, D6A, F84
16	
17	
18	20, 21, 170, 171, 228, 62B, 97B, AC0, AC1, C0B, C4B, F76
19	
1A	

Notice that, as was the case in base 10, when the digits are raised to an odd power, fixed points are possible only when c is a multiple of 3. Restricting patterns are present in assorted bases, and follow the rule below which relates c, q, and b.

Theorem 3. If $b \equiv 1 \pmod{k}$, $k \in \mathbf{Z}^+$, and $a^q{}_i \equiv a_i \pmod{k}$ for $a_i \in \{1,2,...k-1\}$, then there exists a restricting pattern such that if $a = T_{[c,q,b]}(a)$ then c must be a multiple of k.

Proof. If a is a fixed point, then

$$\sum_{i=0}^{n} a_i b^i = c + \sum_{i=0}^{n} a_i{}^q.$$

This means that

$$\sum_{i=0}^{n} a_i b^i \equiv c + \sum_{i=0}^{n} a_i{}^q \quad (mod\ k).$$

Since $b \equiv 1\ (mod\ k)$ and $a_i{}^q \equiv a_i \quad (mod\ k)$, for $a_i \in \{1,2,\dots,k-1\}$,

$$\sum_{i=0}^{n} a_i \equiv c + \sum_{i=0}^{n} a_i \quad (mod\ k)$$
$$0 \equiv c \ \ (mod\ k).$$

Therefore, if $b \equiv 1 \pmod{k}$, $k \in Z^+$, and $a^q{}_i \equiv a_i \pmod{k}$ for $a_i \in \{1,2,...k-1\}$, then there exists a restricting pattern such that if $a = T_{[c,q,b]}(a)$ then c must be a multiple of k.

Conclusion

Under the right circumstances, the fixed points of augmented happy functions of higher power can be predicted and restricted. Currently, research is being conducted on fixed points in other base number systems and what patterns can be observed. Work is also being done on *cycles* for a given c, q, and b. A cycle occurs when, for a given c, q, and b, the augmented happy function of higher power in base b is iteratively applied to a positive integer a and a is the result after some number of iterations. An example of such a cycle is if $c = 2$, $q = 3$, and $b = 10$ then if $a = 67$, we say $67 \rightarrow 561 \rightarrow 344 \rightarrow 157 \rightarrow 471 \rightarrow 410 \rightarrow 67$ since $T_{[2,3,10]}(67) = 561$, $T_{[2,3,10]}(561) = 344$, and so on. Every positive integer will iteratively lead to a fixed point or cycle.

Notes and references

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